Introduction to Game Theory

What is a Game?

A game is a formal representation of a situation in which a number of individuals interact in a setting of *strategic interdependence*. By that, we mean that each individual's welfare depends not only on her own actions but also on the actions of the other individual.

1. The players: Who is involved?
2. The actions and outcomes: For each possible set of actions by the players, what is the outcome of the game?
3. The payoffs: What are the players' preferences (i.e. utility functions) over the possible outcomes?

Example

	A	B
A	3,3	1,4
B	4,1	2,2

Game Theory Notation

- The stage game is represented in standard strategic (normal) form.

- The set of players is denoted by $I = \{1, ..., n\}$.

- Each player $i \in I$ has an *action set* denoted by $\mathcal{A}_i$.

- An *action profile* $a = (a_i, a_{-i})$ consists of the action of player i and the actions of the other players, denoted by $a_{-i} = (a_1, ..., a_{i-1}, a_{i+1}, ..., a_n) \in \mathcal{A}_{-i}$.

- In addition, each player i has a real-valued, stage-game, payoff function $g_i : \mathcal{A} \to \mathbb{R}$, which maps every action profile $a \in \mathcal{A}$ into a payoff for i, where $\mathcal{A}$ denotes the cartesian product of the action spaces $\mathcal{A}_i$, written as $\mathcal{A} \equiv \mathop{\times}\limits_{i=1}^{I} \mathcal{A}_i$.

- In the repeated game with *perfect monitoring*, the stage game in each time period $t = 0, 1, \ldots$ is played with the action profile chosen in period t publicly observed at the end of that period.

- The *history* of play at time t is denoted by $h^t = (a^0, \ldots, a^{t-1}) \in \mathcal{A}^t$, where $a^r = (a_1^r, \ldots, a_n^r)$ denotes the actions taken in period r.

- The set of histories is given by

$$\mathcal{H} = \bigcup_{t=0}^{\infty} \mathcal{A}^t,$$

where we define the initial history to the null set $\mathcal{A}^0 = \{\emptyset\}$.

- A *strategy* $s_i \in S_i$ for player i is, then, a function $s_i : \mathcal{H} \rightarrow \mathcal{A}_i$, where the strategy space of i consists of K_i discrete strategies; that is, $S_i = \{s_i^1, s_i^2, ..., s_i^{K_i}\}$.

- Furthermore, denote a strategy combination of the n players except i by $s_{-i} = (s_1, ..., s_{i-1}, s_{i+1}, ..., s_n)$.

- The set of joint-strategy profiles is denoted by $S = S_1 \times \cdots \times S_n$.

- Each player i has a payoff function $\pi_i : S \rightarrow \mathbb{R}$, which represents the payoff when the joint-strategy profile is played.

Dominant (Pure) Strategies

Consider first the predictions that can be made based on a relatively simple means of comparing a player's possible strategies: that of *dominance*.

	Mum	Confess
Mum	3,3	1,4
Confess	4,1	2,2

- The game above is the famous Prisoner's Dilemma game.
- What will the outcome of the game be?
- There is only one plausible answer: (Confess, Confess)

Playing "Confess" is each player's best strategy *regardless of what the other player does*. This type of strategy is known as a *strictly dominant strategy*.

Dominant (Pure) Strategies

Definition 1. A strategy $s_i \in S_i$ is a **strictly dominant strategy** for player i in game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if $\forall s_i' \neq s_i$, we have

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$$

$\forall s_{-i} \in S_{-i}$.

- In words, a strategy s_i is a strictly dominant strategy for player i if it maximizes uniquely player i's payoff for any strategy that player i's rivals might play.

- The striking aspect of the (Confess, Confess) outcome in the Prisoner's Dilemma is that although it is the one we expect to arise, it is not the best outcome for the players *jointly*; both players would prefer that neither of them confess.

Dominated (Pure) Strategies

- Although it is compelling that players should play strictly dominant strategies if they have them, it is rare for such strategies to exist.
- Even so, we might still be able to use the idea of dominance to eliminate some strategies as possible choices.
- In particular, we should expect that player i will not play *dominated strategies*; those for which there is some alternative strategy that yields him a greater payoff regardless of what the other player will do.

Definition 2. A strategy $s_i \in S_i$ is **strictly dominated** for player i in game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if there exists another strategy $s_i' \in S_i$, such that $\forall s_{-i} \in S_{-i}$, we have

$$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}).$$

Matching Pennies

- Consider the following game.

	Heads	Tails
Heads	1,-1	-1,1
Tails	-1,1	1,-1

- There is no (pure strategy) Nash equilibrium in this game. If we play this game, we should be unpredictable.

Dominant & Dominated Mixed Strategies

The basic definitions of strictly dominant and dominated strategies can be generalized in a straightforward way, when players randomize over their pure strategies.

DEFINITION 3. A strategy $\sigma_i \in \triangle(S_i)$ is a **strictly dominant strategy** for player i in game $\Gamma_N = [I, \{\triangle(S_i)\}, \{u_i(\cdot)\}]$ if $\forall \sigma_i' \neq \sigma_i$, we have

$$u_i(\sigma_i, \sigma_{-i}) > u_i(\sigma_i', \sigma_{-i})$$

$\forall \sigma_{-i} \in \Pi_{j \neq i} \triangle(S_j)$.

DEFINITION 4. A strategy $\sigma_i \in \triangle(S_i)$ is **strictly dominated** for player i in game $\Gamma_N = [I, \{\triangle(S_i)\}, \{u_i(\cdot)\}]$ if there exists another strategy $\sigma_i' \in \triangle(S_i)$, such that $\forall \sigma_{-i} \in \Pi_{j \neq i} \triangle(S_j)$, we have

$$u_i(\sigma_i', \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i}).$$

PROPOSITION 1. Player i's pure strategy $s_i \in S_i$ is strictly dominated in game $\Gamma_N = [I, \{\triangle(S_i)\}, \{u_i(\cdot)\}]$ if and only if there exists another strategy $\sigma_i' \in \triangle(S_i)$, such that

$$u_i(\sigma_i', s_{-i}) > u_i(s_i, s_{-i})$$

$\forall s_{-i} \in S_{-i}$.

The above proposition tells us that to test whether a pure strategy s_i is dominated when randomized play is possible, one needs to check whether any of player i's mixed strategies does better than s_i against every possible profile of pure strategies by i's rivals.

Nash Equilibrium

We present and discuss next the most widely used solution concept in applications of game theory to economics, that of *Nash Equilibrium* [due to Nash (1951)]. For ease of exposition, we initially ignore the possibility that players might randomize over their pure strategies, restricting our attention to game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$.

Definition 5. A strategy profile $s^* = (s_1^*, s_2^*, ..., s_n^*) = (s_i^*, s_{-i}^*)$ constitutes a **Nash Equilibrium** (NE) of game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if $\forall i \in I$,

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$$

$\forall s_i \in S_i$.

In other words, in a Nash equilibrium, each player's strategy choice is a best response to the *actual* strategy choices played by i's rivals.

Nash Equilibrium

A compact restatement of the definition of Nash Equilibrium can be obtained through the introduction of the concept of a player's *best-response correspondence.*

Formally, we say that player i's best-response correspondence $b_i : S_{-i} \rightarrow S_i$ in the game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$, is the correspondence that assigns to each $s_{-i} \in S_{-i}$ the set

$$b_i(s_{-i}) = \{s_i \in S_i | u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}) \ \ \forall s'_i \in S_i \ \}.$$

With this notion, we can restate the definition of a Nash equilibrium as follows: The strategy profile $s^* = (s^*_1, s^*_2, ..., s^*_n) = (s^*_i, s^*_{-i})$ constitutes a Nash Equilibrium of game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if and only if $\forall i \in I$,

$$s^*_i \in b_i(s^*_{-i}).$$

Mixed Nash Equilibrium

It is straightforward to extend the definition of Nash equilibrium to games in which we allow the players to randomize over their pure strategies.

DEFINITION 6. A strategy profile $\sigma^* = (\sigma_1^*, \sigma_2^*, ..., \sigma_n^*) = (\sigma_i^*, \sigma_{-i}^*)$ constitutes a **Nash Equilibrium** of game $\Gamma_N = [I, \{\triangle(S_i)\}, \{u_i(\cdot)\}]$ if $\forall i \in I$,

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*)$$

$\forall \sigma_i \in \triangle(S_i)$.

In a mixed Nash Equilibrium, players assign positive probabilities to their strategies such that the other player is indifferent over his strategies.

Matching Pennies

	Heads	Tails
Heads	1,-1	-1,1
Tails	-1,1	1,-1

- We should randomize (or mix) between strategies so that we do not get exploited.
- But not any randomness will do: suppose Player 1 plays with 75% Heads and with 25% Tails. Then, Player 2 by choosing Tails with 100% chance can get an expected payoff of $0.75 \times 1 + 0.25 \times (-1) = 0.5$. But that cannot happen at equilibrium since Player 1 then wants to play Tails with 100% chance thus deviating from the original mixed strategy.

Matching Pennies

	Heads	Tails
Heads	1,-1	-1,1
Tails	-1,1	1,-1

- Since this game is completely symmetric it is easy to see that at mixed strategy Nash equilibrium both players will choose Heads with 50% chance and Tails with 50% chance.
- In this case the expected payoff to both players is $0.5 \times 1 + 0.5 \times (-1) = 0$ and neither can do better by deviating to another strategy (regardless if it is a mixed strategy or not).

Wimbledon Tennis Serves

- Consider the following game where F stands for forehand and B stands for backhand.

	Serving on F	Serving on B
Expecting on F	90,10	20,80
Expecting on B	30,70	60,40

- The mixed Nash equilibrium strategy is for the receiver to follow $0.3F + 0.7B$ and the server to serve $0.4F + 0.6B$.
- This is the only strategy that cannot be exploited by either player.

Existence of Nash Equilibrium

Does a Nash Equilibrium necessarily exist in a game? Fortunately, the answer turns out to be "yes" under fairly broad circumstances. Here, we describe two of the more important existence results.

Proposition 2. Every game $\Gamma_N = [I, \{\triangle(S_i)\}, \{u_i(\cdot)\}]$ in which the sets $S_1, S_2, ..., S_n$ have a finite number of elements has a mixed Nash Equilibrium.

Thus, for the class of games we have been considering, a Nash Equilibrium *always exists* as long as we are willing to accept equilibria where players randomize.

Existence of Nash Equilibrium

Up to this point, we have focussed on games with finite strategy sets (a finite strategy S_i cannot be convex). However, in economic applications, we frequently encounter games in which players have strategies naturally modeled as continuous variables. This can be helpful for the existence of a pure strategy Nash Equilibrium.

Proposition 3. A Nash Equilibrium exists in game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if $\forall i \in I$,

(i) S_i is a nonemplty, convex and compact subset of some Euclidean space $\mathbb{R}^M$.

(ii) $u_i(s_1, s_2, ..., s_n)$ is continuous in $(s_1, s_2, ..., s_n)$ and quasiconcave in s_i.

Battle of the Sexes

	Opera	Football
Opera	3,2	0,0
Football	0,0	2,3

Stag-Hunt

	Stag	Hare
Stag	3,3	0,2
Hare	2,0	1,1

Chicken

	Swerve	Straight
Swerve	3,3	1,4
Straight	4,1	0,0