Logic and Set Notation

Logic Notation

- p, q, r: statements
- $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$: logical operators
- $\neg p$: not p
- $p \wedge q$: p and q
- $p \vee q$: p or q
- $p \Rightarrow q$: p implies q
- $p \Leftrightarrow q$:p if and only if q

We can build compound sentences using the above notation and then determine their truth or falsity with the use of logical operators.

If p then q

A conditional statement consists of two parts, a hypothesis or antecedent in the "if" clause and a conclusion or consequent in the "then" clause. For instance, "If it rains, then they cancel school."

Given an *if-then* statement "if p, then q," we can create three related statements.

Statement: If p, then q.

Converse: If q, then p.

Inverse: If not p, then not q.

Contrapositive: If not q, then not p.

If the statement is true, then the contrapositive is also logically true. If the converse is true, then the inverse is also logically true.

Example I

Statement: If three angles add up to 180 degrees, then they must form a triangle.

Converse: If three angles form a triangle, then they must add up to 180 degrees.

Inverse: If three angles do not add up to 180 degrees, then they must not form a triangle.

Contrapositive: If three angles do not form a triangle, then they must not add up to 180 degrees.

Example II

Statement: If a quadrilateral is a rectangle, then it has two pairs of parallel sides.

Converse: If a quadrilateral has two pairs of parallel sides, then it is a rectangle. (FALSE!)

Inverse: If a quadrilateral is not a rectangle, then it does not have two pairs of parallel sides. (FALSE!)

Contrapositive: If a quadrilateral does not have two pairs of parallel sides, then it is not a rectangle.

Truth Tables

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

p	$\neg p$
T	F
F	T

Note that in the statement $p \Rightarrow q$, if p is false, then q is *vacuously true*. For example, "all cellphones in the room are turned off" will be true when there are no cellphones in the room.

We can think of the statement $p \Rightarrow q$ as two separate statements. First, *sufficiency*: p is a sufficient condition for q ("if" p then q). Second, *necessity*: q is a necessary condition for p (i.e. "only if" q then p).

Truth Tables

You should make sure that you understand the following truth table which depicts p and q as jointly necessary and sufficient conditions.

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Logical Rules

How do we prove theorems? There are several types of logical rules available.

1. *Simplification*: From $p \wedge q$ we can infer p and we can infer q.
2. *Addition*: From p we can infer $p \vee q$.
3. *Conjunction*: From p and q we can infer $p \wedge q$.
4. *Disjunctive Syllogism*: From $(p \vee q) \wedge \neg p$ we can infer q.
5. *Modus Ponens*: From $(p \Rightarrow q) \wedge p$ we can infer q.
6. *Modus Tollens*: From $(p \Rightarrow q) \wedge \neg q$ we can infer $\neg p$.
7. *Hypothetical Syllogism*: From $(p \Rightarrow q) \wedge (q \Rightarrow r)$ we can infer $(p \Rightarrow r)$.
8. *Constructive Dilemma*: From $(p \Rightarrow q) \wedge (s \Rightarrow t) \wedge (p \vee s)$ we can infer $(q \vee t)$.

Quantifiers

Sometimes we want to state generalities of the form "some element in the set X has a property $p(\cdot)$" and "every element in the set X has property $p(\cdot)$." To do this, we use quantifiers.

Existential: $(\exists x \in X)p(x)$ There exists an x in X such that $p(x)$.

Universal: $(\forall x \in X)p(x)$ For all x in X, $p(x)$.

Set Notation

- X, Y, Z: sets
- x, y, z: elements of sets
- $x \in X$: x is in X or x is an element in X
- $x \notin X$: x is not in X
- $Y \subseteq X$: Y is a subset of X, or Y is included in X
- $Y = X$: Y is equal to X, or $(Y \subseteq X) \wedge (X \subseteq Y)$
- $Y \neq X$: Y is not equal to X, or $\neg(Y = X)$
- $Y \underset{\neq}{\subseteq} X$: Y is a proper subset of X, or $(Y \subseteq X) \wedge (Y \neq X)$
- $\emptyset$: the empty set (set with no elements)

Set Notation

A set can be identified by listing its contents or by specifying a property common to all and only the elements of that set. For example, if X is the set of integers, then we can represent the set consisting of the number 1, 2 and 3 by $\{1, 2, 3\}$ or $\{x \in X | 0 < x < 4\}$, where the second representation is read "the elements x of X such that $0 < x < 4$."

This method of defining sets is useful when the set is large. For example, if X is the set of real numbers, the set of positive real numbers is $\{x \in X | x > 0\}$. We could never list these numbers.

Set Operations

The following are some simple operations on sets. Let X be our "universe of discourse" (that is, everything happens in this set X). Let Y and Z be subsets of X. Then

1. *Intersection*: $Y \cap Z = \{x \in X | (x \in Y) \wedge (x \in Z)\}$.
2. *Union*: $Y \cup Z = \{x \in X | (x \in Y) \vee (x \in Z)\}$.
3. *Complement*: $\bar{Y} = \{x \in X | x \notin Y\}$.
4. *Subtraction*:
   $Y \setminus Z = Y \cap \bar{Z} = \{x \in X | (x \in Y) \wedge (x \notin Z)\}$.

The last operation is also called "the complement of Z relative to Y."

Remarks

Sets are very useful for expressing order. For example, to represent a point on the Cartesian plane by a pair of numbers (one for horizontal and one for vertical position), we have to express order. Whenever order is important, we enclose the elements in parentheses: (5, 2). By convention, it is essential to list 5 before 2 because the point (5, 2) is quite different from the point (2, 5).

The elements do not have to be numbers: we can consider (x, y, z) where $x \in X$, $y \in Y$ and $z \in Z$ with X, Y and Z being any sets at all. For example, let X be the set of numbers of Coke cans, Y be the set of numbers of whiskey shots, and Z be the set of numbers of wine glasses. Then (3, 1, 10) represents 3 Coke cans, 1 whiskey shot, and 10 glasses of wine.

Remarks

When the elements are two, we call the signification of order an "ordered pair." For more elements, we have ordered triples, all the way up to ordered n-tuples. The set of ordered n-tuples from $X_1, X_2, ..., X_n$ is the "n-fold cross product" (sometimes called the Cartesian product) written as $X_1 \times X_2 \times ... \times X_n$.

For example, when $n = 2$ the cross-product of X_1 and X_2 is defined as $X_1 \times X_2 = \{(x_1, x_2) | (x_1 \in X_1) \wedge (x_2 \in X_2)\}$. So,

if $X_1 = \{1, 2, 3\}$ and $X_2 = \{Mike, Suzie, Peter\}$, then the cross-product is $X_1 \times X_2 =$
$\{(1, Mike), (2, Mike), (3, Mike), (1, Suzie), (2, Suzie), (3, Suzie),$
$(1, Peter), (2, Peter), (3, Peter)\}$.

When $X_1 = X_2 = \cdots = X_n = \mathbb{R}$, we refer to their cross product as the "n-dimensional Euclidean space."